2021-2022 May Exam Solutions

SECTION A

1.

E. $u = P\left(\frac{a^3}{4EI} + \frac{b^3}{3EI} + \frac{a^2b}{GJ}\right)$

[2 marks]

SOLUTION Q1

$$U = \frac{a^3P^2}{8EI} + \frac{b^3P^2}{6EI} + \frac{a^2bP^2}{4GJ}$$

Differentiating with respect to load, P, gives:

$$\boldsymbol{u = \frac{\partial U}{\partial P} = P\left(\frac{a^3}{4EI} + \frac{b^3}{3EI} + \frac{a^2b}{2GJ}\right)}$$

2.

B. (i) at $x = 0$, $y = 0$, (ii) at $x = 0$, $\frac{dy}{dx} = 0$, (iii) at $x = \mathrm{L}$, $y = 0$ & (iv) at $x = \mathrm{L}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

[2 marks]

3.

E. No

[2 marks]

SOLUTION Q3

Behaviour is assumed to be all elastic and therefore:

$$\frac{M_y}{I} = \frac{\sigma_y}{y}$$

where M_y is the moment required to cause yielding.

First yield will occur at $y = \pm\frac{d}{2}$, i.e. at the top and bottom edges:

$$\therefore M_y = \frac{\sigma_y \times I}{y} = \frac{\sigma_y \times \left(\frac{bd^3}{12}\right)}{\frac{d}{2}} = \frac{207 \times \left(\frac{200 \times 125^3}{12}\right)}{\frac{100}{2}} = 134{,}765{,}625 \text{ Nmm} = 134.77 \text{ kNm}$$

Since $M < M_y$, yielding does not occur.

4.

C. 107.2 mm

[2 marks]

SOLUTION Q4

2nd moment of area of a solid square cross-section:

$$I = \frac{bd^3}{12} = \frac{2a^4}{12}$$

where $b = 2a$ and $d = a$

$$\therefore a = \sqrt[4]{6I}$$

Substituting in the minimum required 2nd moment of area:

$$a_{min} = \sqrt[4]{6 \times 22{,}000{,}000} = \mathbf{107.2\ mm}$$

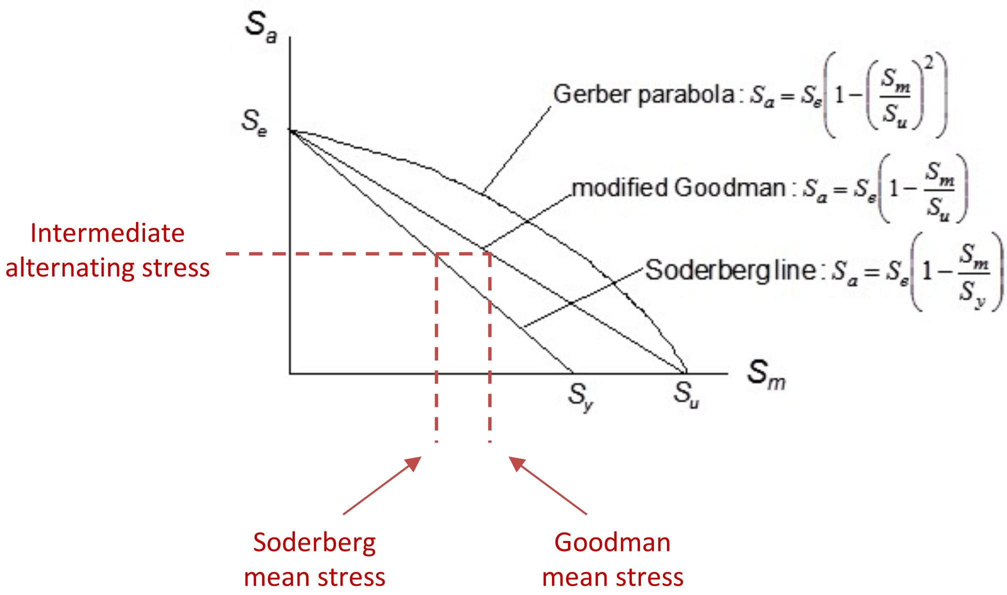
5.

C. The Soberberg line is a better option than the Goodman line or the Gerber curve

[2 marks]

SOLUTION Q5

The figure below shows that at an intermediate alternating stress, the corresponding Soderberg mean stress is less that the corresponding Goodman mean stress. The Soderberg line is therefore more conservative than the Goodman line.



6.

C. $y = \frac{1}{EI}\left(\frac{R_A x^3}{12} + \frac{M_o\langle x-2\rangle^2}{6} - \frac{P_o\langle x-3\rangle^3}{6} + Ax + B\right)$

SOLUTION Q6

$$EI\frac{d^2y}{dx^2} = \frac{R_A}{2}x + \frac{M_o}{3}\langle x-2\rangle^0 - P_o\langle x-3\rangle$$

Integrating this gives:

$$EI\frac{dy}{dx} = \frac{R_A x^2}{4} + \frac{M_o\langle x-2\rangle}{3} - \frac{P_o\langle x-3\rangle^2}{2} + A$$

Integrating again gives:

$$EIy = \frac{R_A x^3}{12} + \frac{M_o\langle x-2\rangle^2}{6} - \frac{P_o\langle x-3\rangle^3}{6} + Ax + B$$

Rearranging this gives:

$$\boldsymbol{y = \frac{1}{EI}\left(\frac{R_A x^3}{12} + \frac{M_o\langle x-2\rangle^2}{6} - \frac{P_o\langle x-3\rangle^3}{6} + Ax + B\right)}$$

7.

A. Kinematic hardening

[2 marks]

8.

D. $U = \int_0^L \frac{T^2}{2JG} ds$

9.

B. 29.2 kN

[2 marks]

SOLUTION Q9

For a fixed-fixed strut under compression:

$$P_{crit} = \frac{4\pi^2 EI}{L^2}$$

Substituting in the values for E, L, b & d:

$$= \frac{4 \times \pi^2 \times 210{,}000 \times \left(\frac{40 \times 20^3}{12}\right)}{2750^2} = 29{,}233.6 \text{ N}$$

$$= \mathbf{29.2\ kN}$$

10.

B. elastic-perfectly-plastic

[2 marks]

11.

D. no stress

[2 marks]

12.

B. ductile material

[2 marks]

13.

A. 140 kPa

[2 marks]

SOLUTION Q13

$$\sigma_r = \tau_{rz} = \tau_{r\theta} = 0$$

$$\sigma_\theta = \frac{PR}{t}$$

$$\sigma_z = \frac{PR}{2t}$$

Both σ_θ and σ_z are tensile stress.

Maximum in-plane tensile stress

$$\sigma_1 = \sigma_\theta = \frac{PR}{t}$$

so:

$$P_{max} = \frac{\sigma_1 t}{R} = \frac{7MPa \times 2mm}{100mm} = 0.14MPa = 140 \text{ kPa}$$

Maximum shear stress:

$$\tau_{max} = R = (\sigma_\theta - \sigma_z)/2 = \frac{PR}{4t}$$

so:

$$P_{max} = \frac{4\tau_{max} t}{R} = \frac{4 \times 2MPa \times 2mm}{100mm} = 0.16MPa = 160 \text{ kPa}$$

When $P = P_{max} = 0.16MPa$:

$$\sigma_1 = \sigma_\theta = \frac{PR}{t} = \frac{0.16MPa \times 100mm}{2mm} = 8 \text{ MPa}$$

which is greater than the allowable tensile stress of 7 MPa, so $P = P_{max} = 0.14 \text{ MPa} = \mathbf{140\ kPa}$.

14.

C. 5.0 MPa

[2 marks]

SOLUTION Q14

$$\sigma_3 = \sigma_r = \tau_{rz} = \tau_{r\theta} = 0$$

$$\sigma_1 = \sigma_\theta = \frac{PR}{t} = \frac{100kPa \times 100mm}{2mm} = 5 \text{ MPa}$$

$$\sigma_2 = \sigma_z = \frac{PR}{2t} = \frac{100kPa \times 100mm}{2 \times 2mm} = 2.5 \text{ MPa}$$

Both σ_θ and σ_z are tensile stress.

So the Tresca equivalent stress on the outer surface is:

$$\sigma_y = \sigma_1 - \sigma_3 = \mathbf{5\ MPa}$$

15.

C. one half of

[2 marks]

16.

E. $2.33 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

[2 marks]

SOLUTION Q16

$$[k_{el}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

where:

$$k = \frac{A \times E}{L} = \frac{0.04m \times 0.05m \times 70 \times 10^9 \text{ N/m}^2}{0.6 \text{ m}} = \mathbf{2.33 \times 10^8\ N/m}$$

17.

A. A

[2 marks]

18.

A. 18.75 MPa

[2 marks]

SOLUTION Q18

The maximum shear stress at the neutral axis is given by:

$$\tau = \frac{SA\bar{y}}{Iz} = \frac{25000 \times \left(\frac{50}{2} - 0\right) \times 40 \times (\frac{50}{2} + 0) \times \frac{1}{2}}{\frac{40 \times 50^3}{12} \times 40} = \mathbf{18.75\ MPa}$$

19.

C. 32.8 MPa

[2 marks]

SOLUTION Q19

For rotating disc, we have:

$$\sigma_r = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

Considering boundary conditions. At r = 0.10 m (inner radius) and r = 0.45 m (outer radius), $\sigma_r = 0$. So we have:

$$A - \frac{B}{0.10^2} - \frac{3+0.25}{8} \times 7200 \times \left(3000 \times \frac{2\pi}{60}\right)^2 \times 0.10^2 = 0$$

$$A - \frac{B}{0.45^2} - \frac{3+0.25}{8} \times 7200 \times \left(3000 \times \frac{2\pi}{60}\right)^2 \times 0.45^2 = 0$$

Solving the equations, we can find A = 61345757.8 and B = 584589.0

So the hoop stress at the external diameter (r = 0.45 m) can be calculated as:

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

$$= 61345757.8 + \frac{584589.0}{0.45^2} - \frac{1+3\times 0.25}{8} \times 7200 \times \left(3000 \times \frac{2\pi}{60}\right)^2 \times 0.45^2$$

$$= 32754748.5\ \text{Pa} = \mathbf{32.75\ MPa}$$

20.

A. Zero

[2 marks]

SECTION B

21.

(a)

Lame's equations:

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$

[1 mark]

Considering boundary conditions:

At $r = \frac{D_i}{2}$, σ_θ is equal to the maximum direct stress (the highest hoop stress will occur at the internal radius), i.e.
$\sigma_\theta = 250MPa = A + \frac{B}{(0.1m)^2}$ (1.1)

[3 marks]

Also at $r = \frac{D_i}{2}$, $\sigma_r = -p_i = -80MPa = A - \frac{B}{(0.1m)^2}$ (1.2)

[2 marks]

And at $r = \frac{D_o}{2}$, $\sigma_r = 0 = A - \frac{B}{\left(\frac{D_o}{2}\right)^2}$ (1.3)

[2 marks]

To find out the minimum possible external diameter D_o, we need to determine A and B first.

Solving the two Equations of 1.1 and 1.2, we can find **A = 85 MPa**, **B = 1.65 MN**.

[2 marks]

Inserting A and B values into Equation 1.3, we can find the minimum possible external diameter is:

$$D_o = 2\sqrt{\frac{B}{A}} = 0.279\ m$$

and the minimum possible wall thickness required is:

$$T = \frac{D_o - D_i}{2} = \frac{0.279 - 0.2}{2} m = \mathbf{0.0395\ m}$$

[2 marks]

(b)

Considering boundary conditions, at $r = \frac{D_o}{2}$, $\sigma_r = \sigma_z = 0$, and $\sigma_\theta = A + \frac{B}{r^2}$

[2 marks]

Also recalling $\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu(\sigma_r + \sigma_z)\right) = \frac{\sigma_\theta}{E} = \frac{u}{r}$, where u is the increase in radius.

[2 marks]

Therefore, at $r = \frac{D_o}{2}$:

$$u = \frac{\sigma_\theta r}{E} = \frac{\sigma_\theta D_o}{2E} = \left(A + \frac{B}{\left(\frac{D_o}{2}\right)^2}\right)\frac{D_o}{2E} = \left(85 + \frac{1.65}{\left(\frac{0.279}{2}\right)^2}\right)\frac{0.279}{2 \times 70{,}000} = 0.338\,mm$$

[2 marks]

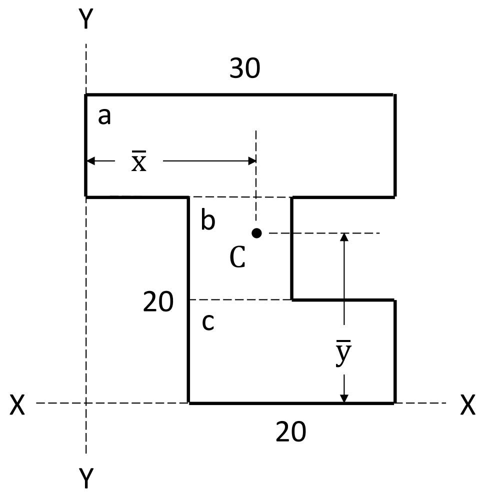
so the increase in outer **diameter** is $2u = \mathbf{0.677\,mm}$.

[2 marks]

22.

(a)

Position of Centroid, C



Total area,

$$A = (30 \times 10)_a + (10 \times 10)_b + (20 \times 10)_c = 600\ \text{mm}^2$$

[1 mark]

Taking moments about YY:

$$A\bar{y} = (30 \times 10 \times 25)_a + (10 \times 10 \times 15)_b + (20 \times 10 \times 5)_c$$

$$\therefore \bar{\boldsymbol{y}} = \mathbf{16.67\ mm}$$

[2 marks]

Similarly, taking moments about XX:

$$A\bar{x} = (30 \times 10 \times 15)_a + (10 \times 10 \times 15)_b + (20 \times 10 \times 20)_c$$

$$\therefore \bar{\boldsymbol{x}} = \mathbf{16.67\ mm}$$

[2 marks]

(b)

Principal 2nd Moments of Area

Using the Parallel Axis Theorem,

$$I_{x'} = (I_x + Ab^2)_a + (I_x + Ab^2)_b + (I_x + Ab^2)_c$$

$$= \left(\frac{30 \times 10^3}{12} + 30 \times 10 \times 8.33^2\right) + \left(\frac{10 \times 10^3}{12} + 10 \times 10 \times -1.67^2\right) + \left(\frac{20 \times 10^3}{12} + 20 \times 10 \times -11.67^2\right)$$

$$= 53{,}333.33 \text{ mm}^4$$

[2 marks]

and,

$$I_{y'} = \left(I_y + Aa^2\right)_a + \left(I_y + Aa^2\right)_b + \left(I_y + Aa^2\right)_c$$

$$= \left(\frac{10 \times 30^3}{12} + 10 \times 30 \times -1.67^2\right) + \left(\frac{10 \times 10^3}{12} + 10 \times 10 \times -1.67^2\right) + \left(\frac{10 \times 20^3}{12} + 10 \times 20 \times 3.33^2\right)$$

$$= 33{,}333.33 \text{ mm}^4$$

[2 marks]

Also,

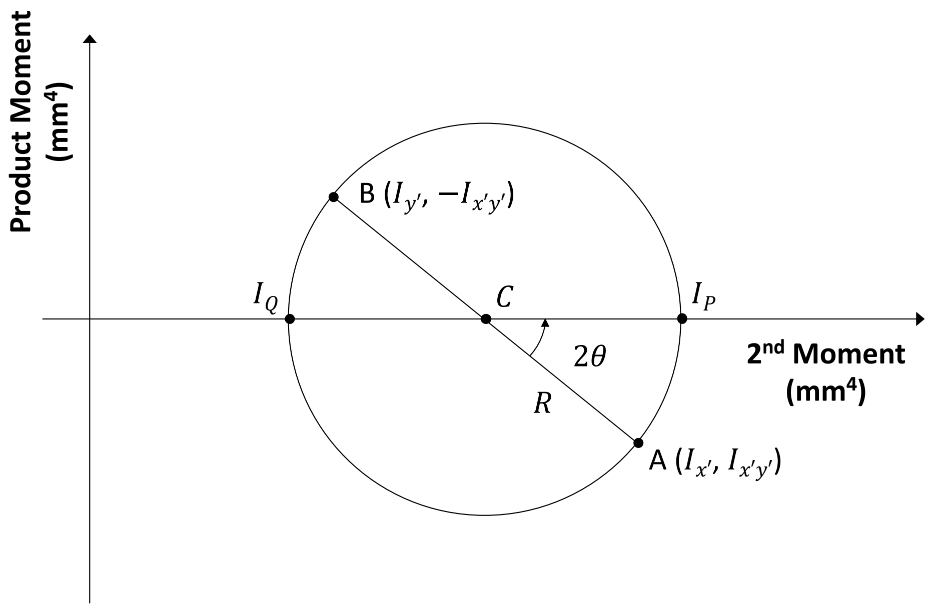
$$I_{x'y'} = \left(I_{xy} + Aab\right)_a + \left(I_{xy} + Aab\right)_b + \left(I_{xy} + Aab\right)_c$$

$$= (0 + 30 \times 10 \times -1.67 \times 8.33) + (0 + 10 \times 10 \times -1.67 \times -1.67) + (0 + 20 \times 10 \times 3.33 \times -11.67)$$

$$= -11{,}666.67 \text{ mm}^4$$

[2 marks]

Mohr's Circle:



$$Centre, C = \frac{I_{x\prime} + I_{y\prime}}{2} = \frac{53{,}333.33 + 33{,}333.33}{2} = 43{,}333.33 \text{ mm}^4$$

[1 mark]

$$Radius, R = \sqrt{\left(\frac{I_{x'} - I_{y'}}{2}\right)^2 + I_{x'y'}{}^2} = \sqrt{\left(\frac{53{,}333.33 - 33{,}333.33}{2}\right)^2 + (-11{,}666.67)^2}$$

$$= 15{,}365.91 \text{ mm}^4$$

[1 mark]

Therefore, the Principal 2nd Moments of Area are:

$$I_P = C + R = 43{,}333.33 + 15{,}365.91$$

$$\therefore \boldsymbol{I_P = 58,699.24 \text{ mm}^4}$$

[1 mark]

and,

$$I_Q = C - R = 43{,}333.33 \ - 15{,}365.91$$

$$\therefore \boldsymbol{I_Q = 27,967.43 \text{ mm}^4}$$

[1 mark]

(c)

Orientation of the Principal Axes with respect to the x-y co-ordinate system

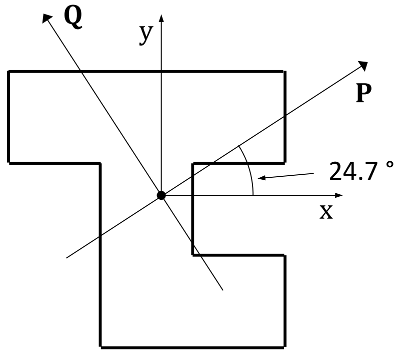
From the Mohr's circle above:

$$sin2\theta = \frac{I_{x'y'}}{R} = \frac{= -11{,}666.67}{15{,}365.91}$$

$$\therefore \theta = -24.7\,°$$

[3 marks]

Therefore, **the Principal Axes are at -32.91° (anti-clockwise) from the x-y axes**, as shown on the diagram below.



[2 marks]

23.

Define rightward force and displacement as positive.

As the displacement of rigid wall 1 is 0, $\mathbf{u_1 = 0}$.

[1 mark]

As spring D only connects rigid bodies 3 and 4, based on the principle of force transmissibility, F_3 = -1,500 N.

[2 marks]

The reaction force on rigid wall 1 has the same amplitude of F_4 but in opposite sign, so F_1 = 1,500 N.

[2 marks]

Considering rigid wall 1 and rigid bodies 2 and 3 only, the force-displacement equations for each element can be written as:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix}$$

[6 marks; 2 marks for each correct element]

Expanding the above equations and inserting F_1, F_3 and u_1 values, we have:

$$\begin{Bmatrix} 1500 \\ F_2 \\ -1500 \end{Bmatrix} = \begin{bmatrix} 400+300 & -400 & -300 \\ -400 & 400+500 & -500 \\ -300 & -500 & 500+300 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

[2 marks]

Considering the first and third rows:

$1500 = 700*0 - 400*u_2 - 300*u_3$

$-1500 = -300*0 - 500*u_2 + 800*u_3$

We can find $\mathbf{u_2 = -1.5957}$ **mm**, $\mathbf{u_3 = -2.8724}$ **mm**.

[2 marks; 1 mark for each correct value]

Considering rigid bodies 3 and 4 only, the force-displacement equations for each element can be written as:

$$\begin{Bmatrix} F_3' \\ F_4' \end{Bmatrix} = \begin{bmatrix} 600 & -600 \\ -600 & 600 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

Inserting F_4' = -1,500 N and u_3 = -2.8724 mm:

-1500 = -600*u_3 + 600*u_4

We can find **u_4 = -5.3724 mm**.

[1 mark for correct value]

So the overall displacement can be written as:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1.5957 \\ -2.8724 \\ -5.3724 \end{Bmatrix}$$

Let's represent the forces in the four springs as F_A, F_B, F_C and F_D, respectively. The forces in the springs can be determined based on Hooke's law:

F_A = k_A * u_3 = 300 * (-2.8724) = -861.6 N (compressive)

F_B = k_B * u_2 = 400 * (-1.5957) = -638.4 N (compressive)

F_C = k_C * (u_3 - u_2) = 500 * (-2.8724+1.5957) = -638.4 N (compressive)

F_D = k_D * (u_4 – u_3) = 600 * (-5.3724+2.8724) = -1500N (compressive)

[4 marks; 1 mark for each correct value]

[2 marks deduction if not stating the spring forces are compressive or tensile]